Equilibrium systems

Sándor Zoltán Németh

Homepage: http://web.mat.bham.ac.uk/S.Z.Nemeth/
Email: nemeths@for.mat.bham.ac.uk

School of Mathematics

UNIVERSITY OF BIRMINGHAM

19th June 2008

Outline

1. Preliminaries
2. A brief overview of Equilibrium systems
    1. Equilibrium systems in general
    2. Particular classes of equilibrium systems
    3. Relations between problems
3. A model of decentralised economy and perfect competition equilibrium
    1. Technical constraints
    2. Budget constraints
    3. Excess supply mapping
    4. Extended Walras law
    5. Walras law
    6. Competitive equilibrium
    7. von Neumann theorem
    8. Gale-Nikaido-Debreu theorem
    9. Equilibrium prices and variational inequalities

Preliminaries

Definition

Let H be a Hilbert space, $K \subseteq H$ a closed convex set and $x \in H$. Then, $\exists! P_K(x) \in K$ such that $\|x - P_K(x)\| \leq \|x - y\|$, $\forall y \in K$. The mapping $P_K : H \to K$, $x \mapsto P_K(x)$ is called the projection mapping onto K.

Proposition

Let H be a Hilbert space, $K \subseteq H$ a closed convex set, $x \in H$ and $z \in K$. Then, $z = P_K(x)$ if and only if $\langle x - z, y - z \rangle \leq 0$, $\forall y \in K$.

Definition

Let H be a Hilbert space. $K \subseteq H$ is called closed convex cone if it is a closed convex set and $\lambda x \in K$, $\forall x \in K$ and $\lambda > 0$. If $K \subseteq H$ is a closed convex cone, then $K^* = \{x \in H : \langle x, y \rangle \geq 0, \ \forall y \in K\}$ is called the dual cone of K.

A brief overview of equilibrium systems

Equilibrium is "everywhere": in economics, physics, engineering, chemistry, biology etc. From the mathematical modelling point of view equilibrium can be described by different types of systems such as fixed point theorems, optimisation problems, variational inequalities, complementarity problems etc.

Equilibrium systems can be studied from several points of view: existence of solutions; existence of nontrivial solutions; number of solutions; properties of the solution set; and the numerical approximation of solutions.

In the first part of my talk after a short description of what is a mathematical equilibrium system in general, I will present several particular classes of equilibrium systems and the relations between then.

Although I will give no details about the theory of these problems, I will try to give hints "between the lines" about what are the main theoretical questions and how can they be handled.

Equilibrium systems in general

Definition

Let K be a set and $f : K \times K \to \mathbb{R}$ with $f(x,x) = 0, \ \forall x \in K$. The equilibrium system $\mathrm{ES}(f,K)$ is the problem of finding an $x \in K$ such that

$$f(x,y) \geq 0, \ \forall y \in K.$$

Particular classes of equilibrium systems

Definition

Let K be a set and $\varphi : K \to \mathbb{R}$. The global optimisation problem $\mathrm{Opt}(\varphi, K)$ is the problem of finding an $x \in K$ such that

$$\varphi(x) \leq \varphi(y), \ \forall y \in K.$$

Definition

Let $(E, \|\cdot\|)$ be a Banach space, $K \subseteq E$ and $T : K \to K$. The fixed point problem $\mathrm{Fix}(T, K)$ is the problem of finding an $x \in K$ such that $x = T(x)$.

Particular classes of equilibrium systems (continued)

Definition

Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space, $K \subseteq E$ and $F : K \to H$. The variational inequality $\mathrm{VI}(F, K)$ is the problem of finding an $x \in K$ such that

$$\langle y - x, F(x)\rangle \geq 0, \ \forall y \in K.$$

Definition

Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space, $K \subseteq E$ a closed convex cone, K^* the dual cone of K and $F : K \to H$. The nonlinear complementarity problem $\mathrm{NCP}(F, K)$ is the problem of finding an $x \in K$ such that $F(x) \in K^*$ and

$$\langle x, F(x)\rangle = 0.$$

Relations between problems

Proposition

Let K *be a set* $\varphi : K \to \mathbb{R}$ *and* $f : K \times K \to \mathbb{R}$ *defined by*

$$f(x, y) = \varphi(y) - \varphi(x).$$

Then,

$$\mathrm{Opt}(\varphi, K) = \mathrm{ES}(f, K).$$

Proposition

Let $(H, \langle \cdot, \cdot \rangle)$ *be a Hilbert space,* $K \subseteq E$, $F : K \to H$ *and* $f : K \times K \to H$ *defined by* $f(x, y) = \langle y - x, F(x) \rangle$. *Then,*

$$\mathrm{VI}(F, K) = \mathrm{ES}(f, K).$$

Relations between problems (continued)

Theorem

Let $(H, \langle \cdot, \cdot \rangle)$ *be a Hilbert space,* $K \subseteq E$ *a closed convex cone,* K^* *the dual cone of* K *and* $F : K \to H$*. Then,*

$$\mathrm{NCP}(F, K) = \mathrm{VI}(F, K).$$

Theorem

Let $K \subseteq \mathbb{R}^n$ *be closed convex and* $\varphi : K \to \mathbb{R}$ *convex and differentiable. Then,*

$$\mathrm{Opt}(\varphi, K) = \mathrm{VI}(\nabla\varphi, K).$$

Relations between problems (continued)

Proposition

Let $(E, \|\cdot\|)$ *be a Banach space,* $K \subseteq H$, $T : K \to K$ *and* $f : K \times K \to \mathbb{R}$ *defined by*

$$f(x,y) = \frac{\|y - T(x)\|^2 - \|2x - y - T(x)\|^2}{4}.$$

Then,

$$\mathrm{Fix}(T,K) = \mathrm{ES}(f,K).$$

In particular, if E *is a Hilbert space, then* $f(x,y) = \langle y - x, x - Tx \rangle$ *and*

$$\mathrm{Fix}(T,K) = \mathrm{VI}(I - T, K).$$

Relations between problems (continued)

Theorem

Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space, $K \subseteq E$ a closed convex set and $F : K \to H$. Then,

$$\mathrm{VI}(F, K) = \mathrm{Fix}(P_K \circ (I - F), K),$$

where P_K is the projection mapping onto K.

Multivalued variational inequalities

Definition

Let $K \subseteq \mathbb{R}^n$, $F : K \multimap \mathbb{R}^n$ such that $F(x) \subseteq \mathbb{R}^n$ is nonempty, convex and compact subset for all $x \in K$. The *multivalued variational inequality* $\mathrm{MVI}(F, K)$ is the problem of finding an

$$x \in K,\ u \in F(x),\ \text{such that } \langle u, y - x\rangle \geq 0,\ \forall y \in K.$$

Proposition

Let

$$f(x, y) = \max_{u \in F(x)} \langle u, y - x\rangle.$$

Then,

$$\mathrm{MVI}(F, K) = \mathrm{ES}(f, K).$$

A model of decentralised economy and perfect competition equilibrium

In the second part of my talk I will show how can a model of descentralised economy be described by a multivalued variational inequality.

In case of Nash equilibrium problems every participant (gambler) estimates the quality of his choice by observing immediately the choice of other participants. Such a direct observation may not be possible in the economic systems. More frequently, the participants have access to a system of prices of the goods that are involved in the model. Each participant undertakes some acts as the response to the observed prices. In the models below, these acts will be formed as the participant's supply at the goods markets. For some goods, the supply may be negative. That reveals the participant's intention to aquire these goods. The supply vector is usually subject to constraints of two types.

Technical constraints

Technical constraints reflect the limited capacities of production or consumption, the limited resources at the participant's disposal (for example, the working hours, the natural resources). Financial constraints presuppose the existence of the circulating money, and model the absence of the budget deficit for every subject of the model. Of course, the budget must take into account both the subject's debts and them possible revenues.

Budget constraints

Consider n types of goods and m economic subjects (agents). Agent i can choose his supply vector x_i in the production set $K^i \subseteq \mathbb{R}^n$ that models the technical constraints of the participant. As the participant of the money circulation, agent i must pay a sum $r_i(p)$ depending on the prices $p = (p_1, \dots, p_n)$ established in the system. Some prices may be zero but the whole vector cannot be zero: $p \neq 0$. The payment $r_i(p)$ may be negative which means that the given agent gets a transfer. It is important to underline that sum $r_i(p)$ does not include the revenue gained by selling the supply x_i at the goods markets.

Budget constraints, continued

This sum is supposed to take into account other relationships existing in the system: property taxes, stocks dividends, pension payments, salary of state employees, etc. The absents of the deficit of the individual budgets is represented by the inequalities

$$\langle p, x_i \rangle \geq r_i(p) \;\; i = 1, \dots, m. \tag{1}$$

These inequalities (1) are referred as the budget constraints. Hence, agent i makes his choice within the subset K^i subject to the corresponding budget constraint (1).

Excess supply mapping

We suppose that, as a result of taking the preference into account, we obtain some multivalued mappings $p \mapsto S_i(p) \subseteq K^i$. For each $x_i \in S_i(p)$ the budget constraint (1) holds. In such models, it is usually assumed that the general price scale does not affect the economic situation. Therefore, the payments $r_i(p)$ are positive homogeneous (of order 1) functions of the prices, that is, $r_i(\alpha p) = \alpha r_i(p)$ for $\alpha > 0$. The choice of the economic subject is also considered as independent of the price scale, as the budget constraints (1) do not depend on the price scale. This means that the supply mappings S_i are positive homogeneous of order 0, i.e., $S_i(\alpha p) = S_i(p)$ for $\alpha > 0$. The mapping S defined by

$$S(p) = \sum_{i=1}^{n} S_i(p) = \left\{ \sum_{i=1}^{n} x_i \ : \ x_i \in S_i(p), \ i = 1, \ldots, m \right\}$$

is called the excess supply mapping.

Extended Walras law

The economic system is very often supposed to be closed in the financial aspect which is expressed by the assumption

$$\sum_{i=1}^{m} r_i(p) = 0 \text{ for all } p. \tag{2}$$

However, in many cases, it is sufficient to demand the absence of outside financial interventions, i.e.,

$$\sum_{i=1}^{m} r_i(p) \geq 0 \text{ for all } p. \tag{3}$$

Together with the budget constraints (1), condition (3) imply the so called Extended Walras law:

$$\langle p, x \rangle \geq 0, \;\; x \in S(p).$$

Walras law

If, we postulate the financial closedness by (2), and the properties of the subsets K^i and the preferences of the economic agents lead to equalities in the budget constraints for the vectors $x_i \in S_i(p)$, then the Walras law is valid:

$$\langle p, x \rangle = 0, \ x \in S(p).$$

Competitive equilibrium

Definition

The prices $\bar{p} \in \mathbb{R}^n_+ \setminus \{0\}$ are referred to as equilibrium ones, if these prices allow a nonnegative total supply, i.e., if there exists a $\bar{x} \in S(\bar{p})$, $\bar{x} \geq 0$. The collection $(\bar{p}; \bar{x}_1, \ldots, \bar{x}_m)$ is called a competitive equilibrium if $\bar{x}_i \in S_i(\bar{p})$ and $\bar{x} := \bar{x}_1 + \cdots + \bar{x}_m \geq 0$.

The attribute "competitive" reflects the fact that the economic subjects make their decision assuming that the prices are constant. Any possibility of individual influence on the prices is rejected.

As the scale of prices does not matter, then together with the equilibrium prices $\bar{p}$, for $\alpha > 0$, prices $\alpha\bar{p}$ will also be equilibrium ones. Therefore, we can suppose that, for a fixed vector of goods $a = (a_1, \ldots, a_n)$, $a_j > 0$, $j = 1, \ldots, n$, the space of prices is

$$P = \{p \in \mathbb{R}^n_+ : \langle p, a \rangle = 1\}.$$

It follows that P is compact and convex. Let $K := K^1 + \cdots + K^m$.

von Neumann theorem

Theorem (von Neumann)

Let $K \subset \mathbb{R}^n$ and $P \subset \mathbb{R}^n$ be nonempty convex compact sets, and F and H be closed subsets of the Cartesian product $K \times P$. If the sets

$$F(x) = \{p \in P : (x, p) \in F\}, \ H(p) = \{x \in K : (x, p) \in H\}$$

are nonempty and convex for any $x \in K$ and $p \in P$, then the intersection $F \cap H$ is not empty.

Gale-Nikaido-Debreu theorem

Lemma

Let the subsets K^i, $i = 1, \ldots, m$, be nonempty compact convex sets, the mappings S_i, $i = 1, \ldots, m$, defined on the nonempty compact convex set P, be closed and every image set $S_i(p)$ be nonempty and convex. Then, the mapping $S = S_1 + \cdots + S_m$ is also closed over P and its image sets $S(p) \subseteq K$ are nonempty and convex subsets of the convex compact set K.

Theorem (Gale-Nikaido-Debreu)

Let the subsets K^i, $i = 1, \ldots, m$, be nonempty compact convex sets. Let S be a mapping closed over the nonempty compact convex set P, and for each $p \in P$, the image set $S(P)$ be a nonempty convex subset of the compact convex subset K. If the extended Walras Law holds, then there exists equilibrium prices, i.e., there are $\bar{p} \in P$ and $\bar{x} \in S(\bar{p})$, $\bar{x} \geq 0$.

Proof of Gale-Nikaido-Debreu theorem

Proof.

Let T be the multivalued mapping defined by

$$x \in K \mapsto T(x) = \{p \in P \,:\, \langle p, x\rangle \leq \langle \tilde{p}, x\rangle \text{ for all } \tilde{p} \in P\}. \qquad (4)$$

Obviously, the mapping T is closed and the sets $T(x)$ are nonempty and convex. By virtue of von Neumann theorem, the graph of the mappings S and T have a common point $(\bar{x}, \bar{p})$ (to be absolutely precise, instead of the graph of T we have to consider the set containing the points $(T(x), x)$). Hence, $\bar{p} \in T(\bar{x})$ and $\bar{x} \in S(\bar{p})$. If we choose $\tilde{p}$ such that $\tilde{p}_i = 0$, $i \neq j$ and $\tilde{p}_j = 1/a_j$, then for $x = \bar{x}$ and $p = \bar{p}$ we obtain $\langle \bar{p}, \bar{x}\rangle \leq \bar{x}_j/a_j$. Since $\bar{x} \in S(\bar{p})$, the extended Walras law implies $\langle \bar{p}, \bar{x}\rangle \geq 0$. As $a_j > 0$, we get $\bar{x}_j \geq 0$.

□

Equilibrium prices and variational inequalities

By analyzing the proof of Gale-Nikaido-Debreu theorem and bearing in mind relation (4), we conclude that the just found equilibrium prices $\bar{p}$ and the corresponding excess supply $\bar{x}$ taken together solve the following multivaled variational inequality problem over the compact set P: Find $\bar{p}$ and $\bar{x}$ such that

$$\bar{p} \in P, \ \bar{x} \in S(\bar{p}), \ \langle \bar{x}, p - \bar{p} \rangle \geq 0 \text{ for all } p \in P.$$